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CONSTRUCTING AN ANALYTICAL SOLUTION FOR LAMB WAVES USING THE COSSERAT CONTINUUM APPROACH

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The problem of propagation of a Lamb elastic wave in a thin plate is considered using the Cosserat continuum model. The deformed state is characterized by independent displacement and rotation vectors. Solutions of the equations of motion are sought in the form of wave packets specified by a Fourier spectrum of an arbitrary shape for three components of the displacement vector and three components of the rotation vector which depend on time, depth, and the longitudinal coordinate. The initial system of equations is split into two systems, one of which describes a Lamb wave and the second corresponds to a transverse wave whose amplitude depends on depth. Analytical solutions in displacements are obtained for the waves of both types. Unlike the solution for Lamb waves, the solution obtained for the transverse wave has no analogs in classical elasticity theory. The solution for the transverse wave is compared with the solution for the Lamb wave.

Key words: *Lamb wave, dispersion, Cosserat medium, analytical solutions.*

Introduction. In the present paper, elastic Lamb waves work are considered using the Cosserat continuum model. A Lamb wave is a normal wave in an elastic wave guide and propagates in thin plates (or films) whose both surfaces are free of loads and whose thickness is of the order of the elastic-wave length. In this case, the plate acts as a wave guide and the displacement vector in the wave has both longitudinal and transverse components, the transverse component being normal to the plate surface.

Since Lamb waves should satisfy not only the elasticity equations but also the boundary conditions on the plate surface, the pattern of motion in these wave and their properties are more complex than those of waves propagating in unbounded solid bodies. This type of waves has been studied well for classical elastic media [1–3].

Lamb waves have found extensive application. In particular, they are used for the overall undestructive control of sheet materials and structures and in signal processing systems (dispersion delay lines). Therefore, in view of the advent of new materials and, accordingly, new theories for their description, an important problem is to extend the well-known classical solutions describing Lamb wave propagation to new models of continuous media. In the present paper, the solution for Lamb waves is extended to the Cosserat elastic linear model.

In the Cosserat continuum theory [1], deformation is described not only by the displacement vector $\boldsymbol{u}$ but also kinematically by the independent vector $\boldsymbol{\omega}$, which characterizes small rotations of particles. In this theory, the stress tensors $\tilde{\sigma}$ and moment stress tensors $\tilde{\mu}$ are asymmetric. The dynamic behavior of the elastic isotropic medium ignoring temperature effects is determined by eight constants: two Lamé constants, four elastic constants describing microstructure, density, and a parameter responsible for the measure of inertia of the medium under rotation (density of the moment of inertia). It is necessary to note that insufficient information on the values of these constants for real structural materials is a major deterrent in the development and application of this theory in practice.

The Cosserat continuum theory predicts a different behavior of waves compared to classical elasticity theory. First, it predicts dispersion for Rayleigh elastic surface waves [4–6] whereas in classical elasticity theory, Rayleigh

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waves do not exhibit dispersion. Second, for the Cosserat model, surface transverse waves propagate with horizontal polarization. In classical elastic theory, the existence of a Love wave as a surface wave is due to the presence of a layer in half-space, and as the thickness of the layer tends to zero, the Love wave becomes a volumetric wave. Kulesh et al. [7] showed that for a Cosserat medium, a horizontally polarized, transverse wave which damps with depth also exists in the absence of a plane layer.

The aforesaid suggests that the differences between Love waves and the classical case are also significant. In [5], a micropolar analog of the classical Lamb dispersion equation is derived and dispersion relations for several wave modes are given in comparison with the classical case. It is shown that in addition to the generalization of the solutions existing in the Rayleigh–Lamb theory, the Cosserat theory contains a solution that has no analog in the classical case.

In the present work, one more solution is obtained which also has no analogs in classical elasticity theory. This solution describes a wave propagating in a plate with one transverse component of the displacement vector and two components of the rotation vector. This wave has an even larger number of modes than a Lamb wave; all these modes exhibit dispersion, and the displacements for all modes depend on depth. Solutions of the equations of motion are obtained for the case of a nonmonochromatic wave and describe the propagation of wave packets specified by a Fourier spectrum of arbitrary shape.

In this paper, we give a solution for the displacement vector, the rotation vector, and the dispersion equation and its numerical solutions for a certain hypothetical set of material parameters.

1. Formulation of the Problem. The basic relations for an elastic Cosserat medium are given by [1]:

— the equations of motion

$$\nabla\cdot\tilde{\sigma}+\boldsymbol{X}=\rho\ddot{\boldsymbol{u}},\qquad \tilde{\sigma}^{\mathrm{t}}:\tilde{\boldsymbol{E}}+\nabla\cdot\tilde{\mu}+\boldsymbol{Y}=j\ddot{\boldsymbol{\omega}}; \tag{1.1}$$

— the geometrical relations

$$\tilde{\gamma}=\nabla\boldsymbol{u}-\tilde{\boldsymbol{E}}\cdot\boldsymbol{\omega},\qquad \tilde{\chi}=\nabla\boldsymbol{\omega}; \tag{1.2}$$

— the physical equations

$$\tilde{\sigma}=2\mu\tilde{\gamma}^{(S)}+2\alpha\tilde{\gamma}^{(A)}+\lambda I_1(\tilde{\gamma})\tilde{e},\qquad \tilde{\mu}=2\gamma\tilde{\chi}^{(S)}+2\varepsilon\tilde{\chi}^{(A)}+\beta I_1(\tilde{\chi})\tilde{e}. \tag{1.3}$$

In view of relations (1.1)–(1.3), the equations of motion for the displacement vector $\boldsymbol{u}$ and $\boldsymbol{\omega}$ and the rotation vector are written as

$$\begin{gathered}(\lambda+2\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}-(\mu+\alpha)\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\omega}+\boldsymbol{X}=\rho\ddot{\boldsymbol{u}},\\ (\beta+2\gamma)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\omega}-(\gamma+\varepsilon)\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\omega}+2\alpha\,\mathrm{rot}\,\boldsymbol{u}-4\alpha\boldsymbol{\omega}+\boldsymbol{Y}=j\ddot{\boldsymbol{\omega}}.\end{gathered} \tag{1.4}$$

In (1.1)–(1.4), $\boldsymbol{X}$ is the specific density vector of the mass forces, $\boldsymbol{Y}$ is the specific density vector of the mass moments, $\boldsymbol{u}$ is the displacement vector, $\boldsymbol{\omega}$ is the rotation vector, $\tilde{\gamma}$ and $\tilde{\chi}$ are the strain and bending-torsion tensors, $\tilde{\sigma}$ and $\tilde{\mu}$ are the stress and moment stress tensors, μ and λ are Lamé constants, α, β, γ, and ε are the physical constants of the material of the elastic Cosserat medium, ρ is the density, j is the density of the moment of inertia (a measure of the inertia of the medium under rotation), $\tilde{\boldsymbol{E}}$ is the third-rank Levi-Civita tensor, $(\,\cdot\,)^{(S)}$ is the symmetrizing operation, $(\,\cdot\,)^{(A)}$ is the alternation operation, $\nabla(\,\cdot\,)$ is the nabla-operator, $I_1(\,\cdot\,)$ the first invariant of the tensor, and $\tilde{e}$ is the unit tensor [8.] In contrast to the classical theory, the tensors $\tilde{\gamma}$ and $\tilde{\sigma}$ are asymmetric.

We consider an elastic layer of thickness $2H$ enclosed between the planes $z=\pm H$. The Cartesian coordinates x and y are directed along the surface, and the z axis is perpendicular to it. The wave is assumed to propagate in the x direction.

In the case of Lamb waves, the boundary conditions define the absence of forces and moments on both surfaces of the layer:

$$\sigma_{zx}|_{z=\pm H}=0,\qquad \sigma_{zy}|_{z=\pm H}=0,\qquad \sigma_{zz}|_{z=\pm H}=0,$$

$$\mu_{zx}|_{z=\pm H}=0,\qquad \mu_{zy}|_{z=\pm H}=0,\qquad \mu_{zz}|_{z=\pm H}=0.$$

2. Constructing the General Solution. Unlike in the well-known papers [2–6], which deal only with monochromatic waves, following the procedure described in [9, 10], we write the general solution of the problem in the form of Fourier integrals with respect to all component of the displacement vector $u_n(x,z,t)$ and the rotation vector $\omega_n(x,z,t)$; this corresponds to the representation of the solution in the form of a wave packet of arbitrary form which is bounded in the time and Fourier spaces:

$$
\begin{aligned}
u_n(x,z,t) &= \int\limits_{-\infty}^{\infty} U_n(z)\,\mathrm{e}^{i(kx+ft)}\,\hat{S}_0(f)\,df, \\
\omega_n(x,z,t) &= \int\limits_{-\infty}^{\infty} W_n(z)\,\mathrm{e}^{i(kx+ft)}\,\hat{S}_0(f)\,df.
\end{aligned} \tag{2.1}
$$

Here $i=\sqrt{-1}$ is imaginary unit, k is the wavenumber, f is the circular frequency (related to the physical frequency p in hertz by the formula $f=2\pi p$), t is time, $U_n(z)$ and $W_n(z)$ are peak functions that depend on depth, and $\hat{S}_0(f)$ is a complex spectral function that corresponds to the Fourier spectrum of the source signal and defines the shape of the wave packet. Here only the material parts of the displacement vector and rotation components have the physical meaning.

Representation (2.1) describes the wave propagating in the negative x direction. The solution for the wave propagating in the positive x direction is obtained similarly by changing the sign of the term kx.

In this case, the fulfillment of the continuous Fourier transform of the equations of motion (1.4) and representations (2.1) is justified. We use direct and Fourier transforms in the form [8]

$$
\hat{S}(f) = \int\limits_{-\infty}^{\infty} S(t)\,\mathrm{e}^{-ift}\,dt, \qquad S(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{S}(f)\,\mathrm{e}^{ift}\,df,
$$

where $\hat{S}(f)$ is a complex function of the Fourier image of the function $S(t)\in L^2(\mathbb{R})$, which is determined on the entire real axis and possesses a finite energetic norm:

$$
\int\limits_{-\infty}^{\infty} |S(t)|^2\,dt < \infty.
$$

This implies the following system of equations for the required components of the displacement and rotation vectors (it is assumed that mass forces and moments are absent):

$$
\begin{aligned}
&(\lambda+2\mu)\,\mathrm{grad}\,\mathrm{div}\,\hat{\boldsymbol{u}} - (\mu+\alpha)\,\mathrm{rot}\,\mathrm{rot}\,\hat{\boldsymbol{u}} + 2\alpha\,\mathrm{rot}\,\hat{\boldsymbol{\omega}} + \rho f^2\hat{\boldsymbol{u}} = \mathbf{0}, \\
&(\beta+2\gamma)\,\mathrm{grad}\,\mathrm{div}\,\hat{\boldsymbol{\omega}} - (\gamma+\varepsilon)\,\mathrm{rot}\,\mathrm{rot}\,\hat{\boldsymbol{\omega}} + 2\alpha\,\mathrm{rot}\,\hat{\boldsymbol{u}} - (4\alpha - jf^2)\hat{\boldsymbol{\omega}} = \mathbf{0}.
\end{aligned} \tag{2.2}
$$

The Fourier transform of representation (2.1) is given by

$$
\begin{aligned}
\hat{\boldsymbol{u}} &= \{U_x(z),U_y(z),U_z(z)\}^{\mathrm{t}}\,\mathrm{e}^{ikx}\,\hat{S}_0(f), \\
\hat{\boldsymbol{\omega}} &= \{W_x(z),W_y(z),W_z(z)\}^{\mathrm{t}}\,\mathrm{e}^{ikx}\,\hat{S}_0(f).
\end{aligned} \tag{2.3}
$$

For the convenience of presentation, we make all quantities dimensionless by using the characteristic size X_0 and the characteristic frequency f_0. In addition, we introduce four dimensionless quantities, one of which depends on the characteristic size:

$$
A = X_0\sqrt{\frac{\mu}{B(\gamma+\varepsilon)}}, \quad B = \frac{\alpha+\mu}{\alpha}, \quad C = \frac{\gamma-\varepsilon}{\gamma+\varepsilon}, \quad F = \frac{B-1}{A^2B}.
$$

Dynamic effects are taken into account by using the dimensionless parameters

$$
C_1^2 = \frac{\lambda+2\mu}{\rho X_0^2 f_0^2}, \quad C_2^2 = \frac{\mu}{\rho X_0^2 f_0^2}, \quad C_3^2 = \frac{B}{B-1}\,C_2^2, \quad C_4^2 = \frac{\gamma+\varepsilon}{jX_0^2 f_0^2}, \quad C_5^2 = \frac{\beta+2\gamma}{jX_0^2 f_0^2}.
$$

Here C_1 and C_2 are analogs of the velocities of the longitudinal and transverse waves and C_4 and C_5 are two additional independent parameters due to the presence of the new material constants of the Cosserat medium; the parameter C_3 is introduced to simplify the presentation.

Applying the method described in detail in [11] to Eqs. (2.2) and (2.3), we obtain the general dimensionless solution of the equations of motion (1.4). It should be noted that unlike in the case of surface waves, a solution for which is considered in [11], in the given case, it is necessary to retain all particular solutions and not only those that damp with depth. Thus, the general solution in displacements is written as

$$
\begin{aligned}
u_x(x,z,t) &= \int\limits_{-\infty}^{\infty} \{D_1 ik\, e^{-\nu_1 z} + D_2\nu_2\, e^{-\nu_2 z} + D_3\nu_3\, e^{-\nu_3 z} + D_4 ik\, e^{\nu_1 z} \\
&\quad - D_5\nu_2\, e^{\nu_2 z} - D_6\nu_3\, e^{\nu_3 z}\}\, e^{i(kx+ft)}\, \hat{S}_0(f)\, df, \\
u_y(x,z,t) &= \frac{F}{2}\int\limits_{-\infty}^{\infty} \Big\{E_2\Big(A_m - \frac{f^2}{C_4^2} + \frac{4}{F}\Big)\, e^{-\xi_2 z} + E_3\Big(A_p - \frac{f^2}{C_4^2} + \frac{4}{F}\Big)\, e^{-\xi_3 z} \\
&\quad + E_5\Big(A_m - \frac{f^2}{C_4^2} + \frac{4}{F}\Big)\, e^{\xi_2 z} + E_6\Big(A_p - \frac{f^2}{C_4^2} + \frac{4}{F}\Big)\, e^{\xi_3 z}\Big\}\, e^{i(kx+ft)}\, \hat{S}_0(f)\, df, \\
u_z(x,z,t) &= \int\limits_{-\infty}^{\infty} \{-D_1\nu_1\, e^{-\nu_1 z} + D_2 ik\, e^{-\nu_2 z} + D_3 ik\, e^{-\nu_3 z} + D_4\nu_1\, e^{\nu_1 z} \\
&\quad + D_5 ik\, e^{\nu_2 z} + D_6 ik\, e^{\nu_3 z}\}\, e^{i(kx+ft)}\, \hat{S}_0(f)\, df, \\
\omega_x(x,z,t) &= \int\limits_{-\infty}^{\infty} \{E_1 ik\, e^{-\xi_1 z} + E_2\xi_2\, e^{-\xi_2 z} + E_3\xi_3\, e^{-\xi_3 z} + E_4 ik\, e^{\xi_1 z} \\
&\quad - E_5\xi_2\, e^{\xi_2 z} - E_6\xi_3\, e^{\xi_3 z}\}\, e^{i(kx+ft)}\, \hat{S}_0(f)\, df, \\
\omega_y(x,z,t) &= \frac{B}{2}\int\limits_{-\infty}^{\infty} \Big\{D_2\Big(A_m - \frac{f^2}{C_3^2}\Big)\, e^{-\nu_2 z} + D_3\Big(A_p - \frac{f^2}{C_3^2}\Big)\, e^{-\nu_3 z} \\
&\quad + D_5\Big(A_m - \frac{f^2}{C_3^2}\Big)\, e^{\nu_2 z} + D_6\Big(A_p - \frac{f^2}{C_3^2}\Big)\, e^{\nu_3 z}\Big\}\, e^{i(kx+ft)}\, \hat{S}_0(f)\, df, \\
\omega_z(x,z,t) &= \int\limits_{-\infty}^{\infty} \{-E_1\xi_1\, e^{-\xi_1 z} + E_2 ik\, e^{-\xi_2 z} + E_3 ik\, e^{-\xi_3 z} + E_4\xi_1\, e^{\xi_1 z} \\
&\quad + E_5 ik\, e^{\xi_2 z} + E_6 ik\, e^{\xi_3 z}\}\, e^{i(kx+ft)}\, \hat{S}_0(f)\, df,
\end{aligned}
\tag{2.4}
$$

where the constants D_i and E_i ($i = 1,\ldots,6$) are determined from the boundary conditions; the exponents ν_m and ξ_m ($m = 1,\ldots,3$) are defined by the relations

$$
\nu_1 = \sqrt{k^2 - \frac{f^2}{C_1^2}}, \quad \xi_1 = \sqrt{k^2 - \frac{f^2}{C_5^2} + \frac{4C_4^2}{FC_5^2}}, \quad \nu_2 = \xi_2 = \sqrt{k^2 - A_m}, \quad \nu_3 = \xi_3 = \sqrt{k^2 - A_p},
$$

$$
A_{p,m} = \frac{C_3^2 + C_4^2}{2C_3^2C_4^2}\, f^2 - 2A^2 \pm \sqrt{\frac{(C_3^2 - C_4^2)^2}{4C_3^4C_4^4}\, f^4 - \frac{2A^2(C_2^2C_3^2 - 2C_3^2C_4^2 + C_2^2C_4^2)}{C_2^2C_3^2C_4^2}\, f^2 + 4A^4}.
$$

3. Rayleigh and Lamb Waves. In the particular case, relations (2.4) adequately describe the well-studied solutions for Rayleigh surface waves in elastic half-space. The given solutions are damping with increasing depth, i.e., the constants at the positive exponents in (2.4) vanish: $D_4 = D_5 = D_6 = 0$ and $E_4 = E_5 = E_6 = 0$. The boundary conditions define the absence of forces and moments on the surface of the half-space ($z = 0$), and in dimensionless variables, they are written as

$$
\begin{aligned}
&\sigma_{zx}|_{z=0} = 0, \qquad \sigma_{zy}|_{z=0} = 0, \qquad \sigma_{zz}|_{z=0} = 0, \\
&\mu_{zx}|_{z=0} = 0, \qquad \mu_{zy}|_{z=0} = 0, \qquad \mu_{zz}|_{z=0} = 0.
\end{aligned}
\tag{3.1}
$$

Substitution of solution (2.4) into boundary conditions (3.1) yields two homogeneous systems of algebraic equations, and from the resolvability condition for these equations, we obtain the following dispersion equations for the two types of waves:

1) for Rayleigh waves with components u_x, u_z, and ω_y [11],

$$\det\left(M_r(\nu_1,\nu_2,\nu_3)\right)=0; \tag{3.2}$$

2) for transverse surface waves with components u_y, ω_x, and ω_z [7],

$$\det\left(M_t(\xi_1,\xi_2,\xi_3)\right)=0. \tag{3.3}$$

In Eqs. (3.2) and (3.3), the matrices $M_r(p_1,p_2,p_3)$ and $M_t(p_1,p_2,p_3)$ are defined by the expressions

$$M_r(p_1,p_2,p_3)=\begin{bmatrix} 2k^2-\dfrac{f^2}{C_2^2} & -2ikp_2 & -2ikp_3 \\ 2ikp_1 & 2k^2-\dfrac{f^2}{C_2^2} & 2k^2-\dfrac{f^2}{C_2^2} \\ 0 & p_2\Big(A_m-\dfrac{f^2}{C_3^2}\Big) & p_3\Big(A_p-\dfrac{f^2}{C_3^2}\Big) \end{bmatrix},$$

$$M_t(p_1,p_2,p_3)=\begin{bmatrix} \dfrac{2ik}{1-B} & p_2\Big(2+\dfrac{A_mC_4^2-f^2}{2A^2C_4^2}\Big) & p_3\Big(2+\dfrac{A_pC_4^2-f^2}{2A^2C_4^2}\Big) \\ ikp_1(1+C) & p_2^2+k^2C & p_3^2+k^2C \\ \Big(\dfrac{C_5^2}{C_4^2}-C-1\Big)k^2-p_1^2\dfrac{C_5^2}{C_4^2} & 2ikp_2(1+C) & 2ikp_3(1+C) \end{bmatrix}.$$

From this it follows that in half-space whose dynamic behavior is described by the Cosserat medium model, in addition to a Rayleigh surface elliptic wave, a surface wave can exist which has one transverse component parallel to the boundary surface and perpendicular to the wave propagation direction. Therefore, this wave can be compared to a Love wave although it is known that in classical elasticity theory, the existence of a Love wave as a surface elastic wave is determined by the presence of a layer in half-space. As the thickness of the layer tends to zero, the Love wave becomes a volumetric wave. Thus, in a Cosserat medium, a qualitatively new wave mode is found which has no analogs in classical elasticity theory.

In constructing the solution describing Lamb wave propagation, we use the layer thickness $X_0=H$ as the characteristic size. Then, the dimensionless boundary conditions become

$$\begin{aligned} &\sigma_{zx}|_{z=\pm1}=0, \qquad \sigma_{zy}|_{z=\pm1}=0, \qquad \sigma_{zz}|_{z=\pm1}=0, \\ &\mu_{zx}|_{z=\pm1}=0, \qquad \mu_{zy}|_{z=\pm1}=0, \qquad \mu_{zz}|_{z=\pm1}=0. \end{aligned} \tag{3.4}$$

As above, substitution of solution (2.4) into boundary conditions (3.4) yields two homogeneous systems of algebraic equations, and the resolvability condition for these systems yields the following dispersion equations for the two types of waves:

1) for Lamb waves with components u_x, u_z, and ω_y,

$$\det\begin{bmatrix} M_r(\nu_1,\nu_2,\nu_3)L(\nu_1,\nu_2,\nu_3) & M_r(-\nu_1,-\nu_2,-\nu_3)L(-\nu_1,-\nu_2,-\nu_3) \\ M_r(\nu_1,\nu_2,\nu_3)L(-\nu_1,-\nu_2,-\nu_3) & M_r(-\nu_1,-\nu_2,-\nu_3)L(\nu_1,\nu_2,\nu_3) \end{bmatrix}=0; \tag{3.5}$$

2) for transverse waves with components u_y, ω_x, and ω_z,

$$\det\begin{bmatrix} M_t(\xi_1,\xi_2,\xi_3)L(\xi_1,\xi_2,\xi_3) & M_t(-\xi_1,-\xi_2,-\xi_3)L(-\xi_1,-\xi_2,-\xi_3) \\ M_t(\xi_1,\xi_2,\xi_3)L(-\xi_1,-\xi_2,-\xi_3) & M_t(-\xi_1,-\xi_2,-\xi_3)L(\xi_1,\xi_2,\xi_3) \end{bmatrix}=0. \tag{3.6}$$

In Eqs. (3.5) and (3.6), $L(p_1,p_2,p_3)$ is a diagonal matrix of the form

$$L(p_1,p_2,p_3)=\begin{bmatrix} e^{-p_1} & 0 & 0 \\ 0 & e^{-p_2} & 0 \\ 0 & 0 & e^{-p_3} \end{bmatrix}.$$

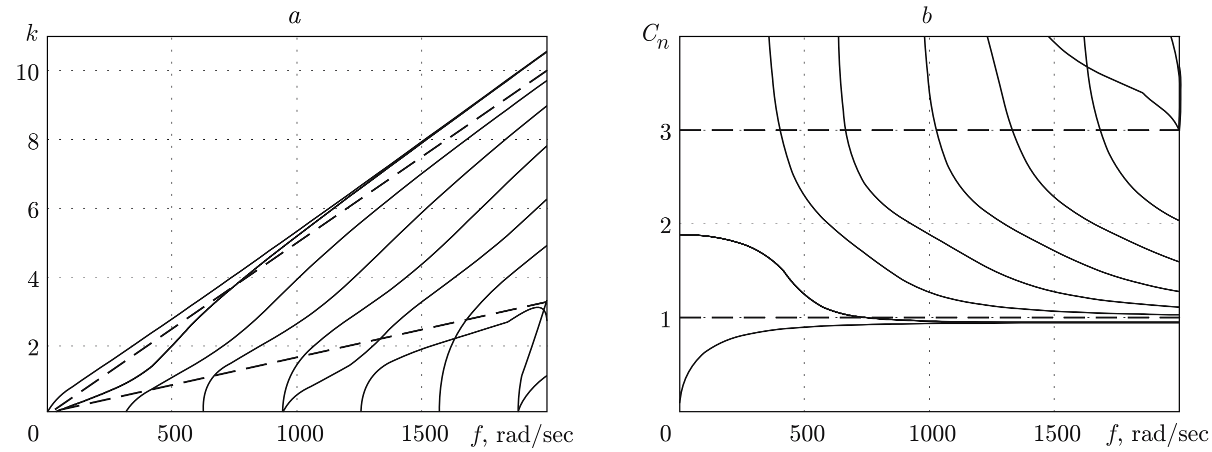

Fig. 1. Wavenumber (a) and normalized phase velocity (b) versus frequency for Lamb waves in a classical elastic medium.

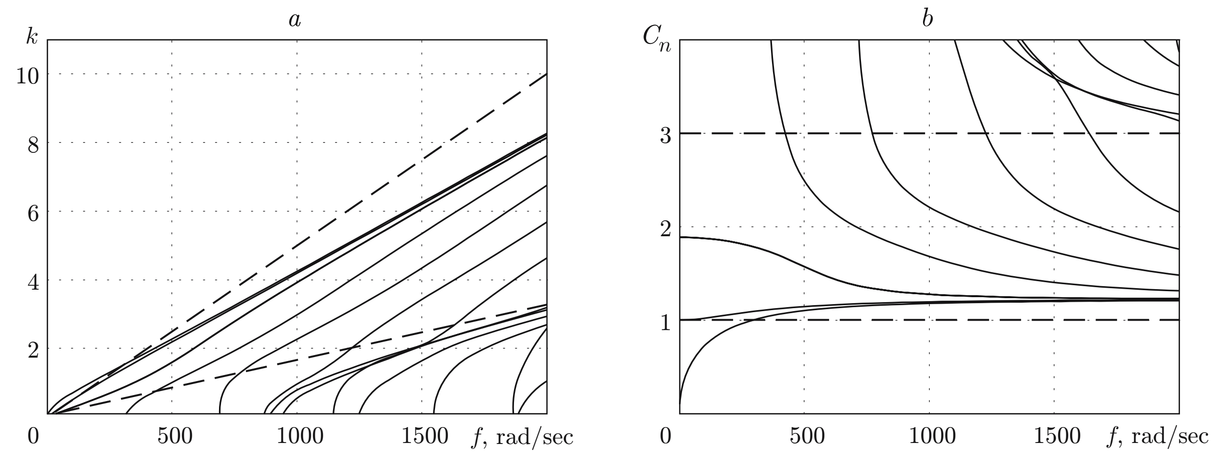

Fig. 2. Wavenumber (a) and normalized phase velocity (b) versus frequency for Lamb wave in a Cosserat medium.

This solution suggests that in a layer whose dynamic behavior is described by the Cosserat medium model, in addition to a Lamb wave there is a fundamentally new wave mode which has one transverse displacement component and whose amplitude varies with depth. This wave mode is absent in classical elasticity theory.

4. Solution of the Dispersion Equations. The solutions of Eqs. (3.2) and (3.3) are analyzed in [7]. The dispersion curves which are solutions of Eqs. (3.5) and (3.6) are given below. For numerical analysis, we use the following values of the material parameters: $\lambda = 2.8 \cdot 10^{10}$ N/m^2, $\mu = 4 \cdot 10^9$ N/m^2, $\rho = 10^5$ kg/m^3, $\alpha = 2 \cdot 10^9$ N/m^2, $\beta = 10^8$ N, $\gamma = 1.936 \cdot 10^8$ N, $\varepsilon = 3.046 \cdot 10^9$ N, and $j = 10^4$ kg/m.

Curves of wavenumbers and normalized phase velocities $C_n(f) = C_p(f)/C_2 = f/(C_2 k(f))$ for a Lamb waves in a classical elastic medium [2] are shown in Fig. 1. (In Figs. 1–3, the dashed curves correspond to longitudinal and transverse waves.) Similar dependences for a Cosserat medium which correspond to solution (3.5) are given in Fig. 2. Thus, asymmetric theory predicts new wave modes for Lamb wave propagation in a plate. A mathematical foundation for this is given in [5]. The data in Fig. 3 correspond to the solution of the dispersion equation (3.6) for a transverse wave with components u_y, ω_x, and ω_z which exhibits dispersion, has many wave modes, and whose amplitude depends on depth.

Conclusions. The finding of the present work is as follows. In a thin plate whose dynamic behavior is described by the Cosserat medium model, in addition to a Lamb wave, there may be a wave which has one transverse component parallel to the boundary surface and perpendicular to the wave propagation direction. Thus, in a Cosserat medium, a qualitatively new wave mode is found that has no analogs in classical elasticity theory.

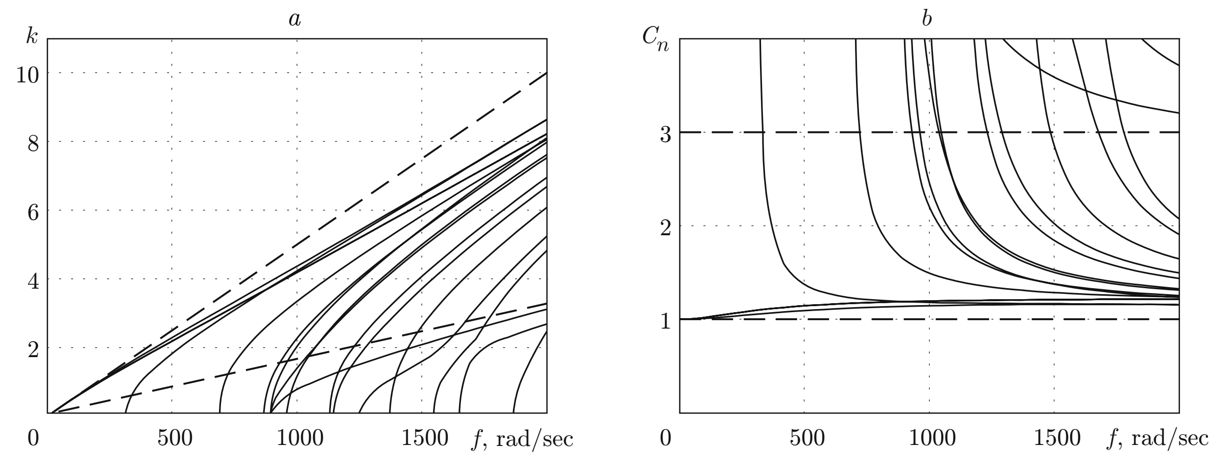

Fig. 3. Wavenumber (a) and normalized phase velocity (b) versus frequency for a transverse wave in a Cosserat medium.

The results of this work may be useful in preparing, carrying out, and interpreting dynamic (wave) experiments designed to determine the role of asymmetric elasticity theory in continuum mechanics and to identify the material parameters of a Cosserat medium.

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